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The Abel-Steffensen inequality in higher dimensions

CONSTANTIN P. NICULESCU

Dedicated to Professor Romulus Cristescu on the occasion of his 90th birthday

ABSTRACT. The Abel-Steffensen inequality is extended to the context of several variables. Applications to Fourier analysis and Riemann-Stieltjes integration are included.

1. INTRODUCTION

Abel's partial summation formula is a polynomial identity that asserts that every pair of families $(a_k)_{k=1}^n$ and $(b_k)_{k=1}^n$ of complex numbers verifies

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} \left[(a_k - a_{k+1}) \left(\sum_{j=1}^{k} b_j \right) \right] + a_n \left(\sum_{j=1}^{n} b_j \right).$$

An immediate consequence is the following inequality, known as Abel's inequality:

$$\left| \sum_{k=1}^{n} a_k b_k \right| \leq a_1 \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} b_k \right|.$$

whenever $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $b_1, b_2, ..., b_n \in \mathbb{C}$. It is this inequality that allowed Abel [1] to prove his well known test of convergence for signed series. See Choudary and Niculescu [4], for a complete account concerning the contribution of Abel to this matter.

One hundred years later, Steffensen [14] noticed another useful consequence of Abel's partial summation formula, that will be referred to as the Abel-Steffensen inequality: if

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \text{ and } \sum_{k=1}^{j} b_k \geq 0 \text{ for all } j \in \{1, 2, ..., n\},$$

then

$$\sum_{k=1}^{n} a_k b_k \geq 0.$$

Using this inequality, Steffensen succeeded to extend Jensen's inequality for convex functions beyond the framework of positive measures.

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As was noticed by Hardy [5], Abel's partial summation formula can be easily extended to the case of double series as follows:

$$(H)\qquad \sum_{i=1}^{p}\sum_{j=1}^{q} a_{ij}u_{ij} = \sum_{i=1}^{p-1}\sum_{j=1}^{q-1}\Delta_{ij}\left(\sum_{k=1}^{i}\sum_{l=1}^{j}u_{kl}\right) + \sum_{i=1}^{p-1}\Delta_{iq}\left(\sum_{k=1}^{i}\sum_{l=1}^{q}u_{kl}\right)$$
$$+\sum_{j=1}^{q-1}\Delta_{pj}\left(\sum_{k=1}^{p}\sum_{l=1}^{j}u_{kl}\right) + a_{pq}\left(\sum_{k=1}^{p}\sum_{l=1}^{q}u_{kl}\right),$$

where

$$\Delta_{ij} = \begin{cases} a_{i,j} - a_{i+1,j} - a_{i,j+1} + a_{i+1,j+1} & \text{if } 1\le i<p,\ 1\le j<q \\ a_{i,q} - a_{i+1,q} & \text{if } 1\le i<p \text{ and } j=q \\ a_{p,j} - a_{p,j+1} & \text{if } i=p \text{ and } 1\le j<q. \end{cases}$$

An immediate consequence is the Abel-Steffensen inequality for double sums:

Theorem 1.1. *If $\mathbf{a} = (a_{ij})_{i,j}$ and $\mathbf{u} = (u_{ij})_{i,j}$ are two double sequences of real numbers such that*

$$a_{ij} \ge 0,\ \Delta_{ij} \ge 0 \text{ and } \sum_{k=1}^{i}\sum_{l=1}^{j}u_{kl} \ge 0$$

for all $i \in \{1,2,...,p\}$ and $j \in \{1,2,...,q\}$, then

$$\sum_{i=1}^{p}\sum_{j=1}^{q} a_{ij}u_{ij} \ge 0.$$

The property $\Delta_{ij} \ge 0$ of the double sequence $\mathbf{a} = (a_{ij})_{i,j}$ represents a 2-dimensional analogue of the usual condition of downward monotonicity for real sequences. In what follows we will refer to it as the property of $2d$-monotone decreasing.

Hardy used the formula (H) to extend the Abel-Dirichlet criterion of convergence to the case of multiple series such as

$$\sum_{m,n\ge 1}\frac{\sin(mx+ny)}{\sqrt{m+n}}.$$

His argument (inspired by the 1-dimensional case) combined the boundedness of the partial sums of the trigonometric series $\sum_{m,n\ge1}\sin(mx+ny)$ with the fact that the double sequence $\left((m+n)^{-1/2}\right)_{m,n\ge1}$ is $2d$-monotone decreasing (see Lemma 2.1 below).

In 1986, David and Jonathan Borwein [3] sketched the necessary formalism for defining the general concept of Nd-monotonicity (for sequences $f:\mathbb{N}^N\to\mathbb{R}$) and established the extension of Leibniz test of convergence to the framework of alternating multiple series. Their work was motivated by the case of a chemical lattice sum,

$$\sum_{m,n,p\ge1}\frac{(-1)^{m+n+p}}{\sqrt{m^2+n^2+p^2}},$$

representing the so called Madelung's constant for sodium chloride.

The aim of this paper is to prove an integral analogue of Theorem 1.1, in the setting of $2d$-monotone functions and to outline several consequences of it to Fourier analysis and Riemann-Stieltjes integral of several variables. See Theorem 3.2, Section 3. For reader's convenience we summarized in Section 2 the main features of 2d-monotonicity, the natural analogue of monotonicity for functions of two variables.

2. $2d$-MONOTONE FUNCTIONS

Let $A = I \times J$ be a rectangle in $\mathbb{R}^2$ (whose sides I and J are intervals parallel to the coordinate axes).

Definition 2.1. A function $f : I \times J \to \mathbb{R}$ is called $2d$*-monotone* if the f-measure of every compact subinterval $A = [a, b] \times [c, d] \subset I \times J$ is nonnegative, that is,

$$[f; A] = f(a, c) - f(a, d) - f(b, c) + f(b, d) \geq 0. \tag{2.1}$$

The function f is called $2d$*-monotone increasing/decreasing* if f is $2d$-monotone and also increasing/decreasing in each variable (when the other one is kept fixed). The function f is called $2d$*-alternating* if $-f$ is $2d$-monotone.

The terminology introduced by Definition 1 is motivated by the fact that any $2d$-monotone function f verifies an inequality of the type

$$[f; A] \leq [f; B] \tag{2.2}$$

for all compact subintervals $A, B \subset I \times J$, with $A \subset B$.

Remark 2.1. One can easily show that f is $2d$-monotone if and only if for every interval $A = [a, b] \times [c, d] \subset I \times J$ the function $\Delta_a^b f(x, y) = f(b, y) - f(a, y)$ is increasing on $[c, d]$ and the function $\Delta_c^d f(x, y) = f(x, d) - f(x, c)$ is increasing on $[a, b]$. Notice that

$$[f; A] = \Delta_c^d \Delta_a^b f = \Delta_a^b \Delta_c^d f.$$

Example 2.1. The product $f(x)g(y)$ of any pair of increasing/decreasing functions $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ is $2d$-monotone increasing on $I \times J$; if the functions have opposite monotonicity then their product is $2d$-alternating. Finite linear combinations $\sum_k c_k f_k(x) g_k(y)$, (with positive coefficients) of such functions have the same nature.

Calculus offers the following useful criterion of $2d$-monotonicity:

Lemma 2.1. *Assume that f is a continuously differentiable function on a rectangle $I \times J$ which admits a continuous second order partial derivative $\frac{\partial^2 f}{\partial x \partial y}$. Then f is $2d$-monotone if and only if*

$$\frac{\partial^2 f}{\partial x \partial y} \geq 0. \tag{2.3}$$

Under the assumptions of Lemma 2.1, one can prove that the partial derivative $\frac{\partial^2 f}{\partial y \partial x}$ also exists and equals $\frac{\partial^2 f}{\partial x \partial y}$. See [2].

Proof. The implication (2.3) $\Rightarrow$ (2.1) follows by integration and a remark due to Aksoy and Martelli [2] that asserts that the continuous differentiability of f together with the existence of a continuous mixed derivative $\frac{\partial^2 f}{\partial x \partial y}$ imply the existence of the other mixed derivative and also their equality. Indeed, for every compact subinterval $A = [a, b] \times [c, d] \subset I \times J$ we have

$$\begin{aligned} 0 &\leq \int_a^b \int_c^d \frac{\partial^2 f}{\partial x \partial y} dy dx \\ &= \int_a^b \left[\frac{\partial f}{\partial x}(x, d) - \frac{\partial f}{\partial x}(x, c) \right] dx = f(b, d) - f(a, d) - f(b, c) + f(a, c). \end{aligned}$$

Conversely, if the condition (2.1) is fulfilled, then by the mean-value theorem we get

$$\frac{\partial^2 f}{\partial x \partial y}(u, v) = \lim_{h \to 0} \frac{f(u, v) - f(u, v + h) - f(u + h, v) + f(u + h, v + h)}{h^2} \geq 0.$$

$\square$

According to Lemma 1, the following functions are $2d$-monotone,

$$\begin{aligned} E(x, y) &= x^2 + y^2 \quad (\text{on } \mathbb{R}^2), \\ C(x, y) &= (e^x + e^y - 1)^{-1} \quad (\text{on } \mathbb{R}_+^2) \\ \Pi(x, y) &= xy \quad (\text{on } \mathbb{R}^2), \end{aligned}$$

while $h(x, y) = \log(x^n + y^n)$ is $2d$-alternating on $(0, \infty) \times (0, \infty)$ whenever $n \geq 1$.

The convex functions give rise to $2d$-monotone functions in a natural manner.

Example 2.2. Since any convex function $F : \mathbb{R} \to \mathbb{R}$ verifies the inequality

$$(z - y)F(x) - (z - x)F(y) + (y - x)F(z) \geq 0$$

for all $x < y < z$ (see [8], Lemma *1.3.2*), one can attach to it a $2d$-monotone function on $\mathbb{R} \times \mathbb{R}$ via the formula $f(x, y) = -F(x - y)$. In a similar way, if $F : [0, \infty) \to \mathbb{R}$ is a convex function and $\lambda > 0$, then $f(x, y) = F(\lambda (x + y))$ is a $2d$-monotone function on $[0, \infty) \times [0, \infty)$.

Example 2.3. If $F : \mathbb{R} \to \mathbb{R}$ is a convex function, then $f(x,y) = \frac{F(x)+F(y)}{2} - F\left(\frac{x+y}{2}\right)$ is $2d$-alternating on $\mathbb{R} \times \mathbb{R}$.

Probability and statistics provide yet another major source of 2d-monotone functions: the functions that couple multivariate distribution functions to their one-dimensional margin-al distribution functions, called copulas by Sklar.

Definition 2.2. *(Hoeffding, Fréchet and Sklar* [12]*)* A copula is a function $C(x,y) : [0,1] \times [0,1] \to [0,1]$ that has the following two properties: (a) (Boundary Conditions) $C(x,0) = C(0,y) = 0$ and $C(x,1) = x$, $C(1,y) = y$ for all $x, y \in [0,1]$; (b) ($2d$-Monotonicity) If $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq y_1 \leq y_2 \leq 1$, then

$$C(x_2,y_2) - C(x_1,y_2) - C(x_2,y_1) + C(x_1,y_1) \geq 0.$$

An important class of copulas is that of *Archimedean copulas.* They are constructed through a continuous, strictly decreasing and convex generator φ as

$$A_\varphi(x,y) = \varphi^{-1}(\varphi(x) + \varphi(y)).$$

Archimedean copulas represent examples of $2d$-monotone increasing functions.

It is worth noticing that copulas are in a one-to-one correspondence with Markov operators $T : L^\infty([0,1]) \to L^\infty([0,1])$ (as well as with doubly stochastic measures on the unit square $[0,1] \times [0,1]$). Details can be found in the book of Nelsen [7].

3. The extension of Abel-Steffensen inequality

The main feature of a positive measure is that the integral of a nonnegative function is a nonnegative number. However, this property still works for some signed measures when restricted to suitable subcones of the cone of positive integrable functions. This phenomenon, first illustrated by the Abel-Steffensen inequality in dimension 1, received a great deal of attention during the last two decades. See [9] and the references therein. The following result provides an integral analogue of Abel-Steffensen inequality in dimension 2. Its argument can be extended in an evident manner to cover all dimensions, but details are too technical to be included here.

Theorem 3.2. *Suppose that f and w are two real-valued functions defined on $[a,b] \times [c,d]$ that fulfill the following conditions: (i) f is continuously differentiable and $2d$-monotone decreasing; (ii) w is integrable and $W(x,y) = \int_a^x \int_c^y w(s,t)dtds \geq 0$ for all $(x,y) \in [a,b] \times [c,d]$. Then*

$$\int_a^b \int_c^d f(x,y)w(x,y)dydx \geq f(b,d)W(b,d).$$

Proof. Using the technique of Dirac sequences of approximating continuous and integrable functions f by infinitely differentiable functions we may restrict ourselves to the case where the hypothesis (i) is replaced by the following stronger condition:

$$(i') \quad \left\{ \begin{array}{c} f \text{ is nonnegative and twice continuously differentiable} \\ \frac{\partial f(x,d)}{\partial x} \leq 0, \ \frac{\partial f(b,y)}{\partial y} \leq 0 \text{ and } \frac{\partial^2 f(x,y)}{\partial x \partial y} \geq 0 \end{array} \right.$$

for all $(x,y) \in [a,b] \times [c,d]$. This can be done by replacing the initial function f with a convolution product $f_n = \rho_n * f$, where

$$\rho_n(x,y) = n^2 \rho(nx, ny) \text{ for } n \geq 1,$$

$$\rho(x,y) = \left\{ \begin{array}{ll} c^{-1} \exp\left(1/\left(x^2+y^2-1\right)\right) & \text{if } x^2+y^2<1 \\ 0 & \text{if } x^2+y^2 \geq 1 \end{array} \right.$$

and

$$c = \iint_{x^2+y^2<1} \exp\left(1/\left(x^2+y^2-1\right)\right) dxdy.$$

For details, see Choudary and Niculescu [4], Theorem 11.7.6 (c), p. 404. Now the proof can be completed by using an identity proved by W. H. Young in [16], p. 38 (and rediscovered later by Pečarić [10]). We provide here a shorter argument for Young's identity, based on repeated use of integration by parts for absolutely continuous functions. Details concerning the formula of integration by parts can be found in the monograph of Hewitt and Stromberg [6], Theorem 18.19, p. 287. Using the aforementioned formula of integration by parts and Fubini's theorem we have

$$\begin{aligned} I &= \int_a^b \int_c^d f(x,y) w(x,y) dy dx \\ &= \int_a^b \left[f(x,d) \int_c^d w(x,y) dy - \int_c^d \left(\int_c^y w(x,t) dt \right) \frac{\partial f(x,y)}{\partial y} dy \right] dx \\ &= \int_a^b \left(f(x,d) \int_c^d w(x,y) dy \right) dx - \int_a^b \left[\int_c^d \left(\int_c^y w(x,t) dt \right) \frac{\partial f(x,y)}{\partial y} dy \right] dx \\ &= \int_c^d \left(\int_a^b f(x,d) w(x,y) dx \right) dy - \int_c^d \left[\int_a^b \left(\int_c^y w(x,t) dt dx \right) \frac{\partial f(x,y)}{\partial y} dx \right] dy, \end{aligned}$$

so that

$$\begin{aligned} I = \int_c^d & \left[f(b,d) \int_a^b w(x,y) dx - \int_a^b \frac{\partial f(x,d)}{\partial x} \left(\int_a^x w(s,y) ds \right) dx \right] dy \\ & - \int_c^d \left[\left(\int_a^b \int_c^y w(x,t) dt dx \right) \frac{\partial f(b,y)}{\partial y} \right] dy \\ & + \int_c^d \int_a^b \left[\left(\int_a^x \int_c^y w(s,t) dt ds \right) \frac{\partial^2 f(x,y)}{\partial x \partial y} dx \right] dy \end{aligned}$$

$$= f(b,d)W(b,d) - \int_a^b \frac{\partial f(x,d)}{\partial x} W(x,d)dx \\ - \int_c^d W(b,y)\frac{\partial f(b,y)}{\partial y}dy + \int_a^b \int_c^d W(x,y)\frac{\partial^2 f(x,y)}{\partial x \partial y}dydx.$$

Now the conclusion follows easily by taking into account the hypotheses (i') and (ii). $\square$

Remark 3.2. An inspection of the argument of Theorem 3.2 shows that the conclusion still works if the hypothesis (i) is replaced by the following one: (i^*) f is nonnegative, continuous, $2d$-monotone decreasing and admits a representation of the form

$$(AC) \qquad f(x,y) = f(a,c) + \int_a^x g_1(s)ds + \int_c^y g_2(t)dt + \int_a^x \int_c^y g(s,t)dtds,$$

for suitable $g_1 \in L^1([a,b])$, $g_2 \in L^1([c,d])$ and $g \in L^1([a,b] \times [c,d])$. The existence of the representation (AC) is equivalent to the condition of absolute continuity in the sense of Carathéodory. See Šremr [13].

Using a similar idea, one can prove the following companion of Theorem 3.2:

Theorem 3.3. *Suppose that f and w are two real-valued functions defined on $[a,b] \times [c,d]$ that fulfill the following conditions: (i) f is nonnegative, continuously differentiable and $2d$-monotone increasing; (ii) w is integrable and $\widetilde{W}(x,y) = \int_x^b \int_y^d w(s,t)dtds \geq 0$ for all $(x,y) \in [a,b]\times[c,d]$. Then*

$$\int_a^b \int_c^d f(x,y)w(x,y)dydx \geq f(a,c)\widetilde{W}(a,c).$$

As above, the continuous differentiability of f can be relaxed to absolute continuity.

The proof of Theorem 3.3 parallels that of Theorem 3.2, using the following variant of Young's identity:

$$\int_a^b \int_c^d f(x,y)w(x,y)dydx = f(a,c)\widetilde{W}(a,c) + \int_a^b \widetilde{W}(x,c)\frac{\partial f(x,c)}{\partial x}dx \\ + \int_c^d \widetilde{W}(a,y)\frac{\partial f(a,y)}{\partial y}dy + \int_a^b \int_c^d \widetilde{W}(x,y)\frac{\partial^2 f(x,y)}{\partial x \partial y}dydx.$$

A well known classical result concerning the Fourier coefficients of a convex function of one variable asserts that

$$\int_0^{2\pi} f(x)\cos nx dx \geq 0 \quad \text{for all positive integers } n.$$

See [8], Exercise 7, p. 26. As a consequence, $\int_0^{2\pi} \int_0^{2\pi} f(x,y)\cos mx \cos ny dxdy \geq 0$ for all convex functions $f : [0,2\pi] \times [0,2\pi] \to \mathbb{R}$ and all positive integers m and n. When cosines are replaced by sines, the corresponding inequality may fail even in the one variable case.

For example, $\int_0^{2\pi} x^2 \sin nx dx < 0$ for all positive integers n. However, based on Theorem 3.2 and Theorem 3.3 we will prove the following result:

Proposition 3.1. *For all monotone and convex functions* $f : [0, 4\pi] \to \mathbb{R}$ *and all positive integers* m *and* n,

$$\int_0^{2\pi} \int_0^{2\pi} f(s+t) \sin ms \sin nt ds dt \geq 0.$$

Proof. Adding to f a suitable constant, we may assume that $f \geq 0$. Then notice that

$$W(x,y) = \int_0^x \int_0^y \sin ms \sin nt ds dt = \frac{(1 - \cos mx)(1 - \cos ny)}{mn} \geq 0$$

and

$$\tilde{W}(x,y) = \int_x^{2\pi} \int_y^{2\pi} \sin ms \sin nt ds dt = \frac{(1 - \cos mx)(1 - \cos ny)}{mn} \geq 0$$

for all $x, y \in [0, 2\pi]$. The conclusion of the corollary follows from Theorem 3.2 when f is decreasing, and from Theorem 3.3 when f is increasing. $\square$

Remark 3.3. An inspection of the proof of Theorem *3.2* easily shows that the conclusion remains valid when the hypotheses $(i)\&(ii)$ are replaced by the following ones: (i'') f is continuously differentiable, $2d$-alternating and its partial derivatives $\frac{\partial f(x,d)}{\partial x}$ and $\frac{\partial f(b,y)}{\partial y}$ are nonnegative; (ii'') w is integrable and $W(x,y) = \int_a^x \int_c^y w(s,t) dt ds \leq 0$ for all $(x,y) \in [a,b] \times [c,d]$. An illustration is offered by the functions $f(x,y) = \ln\left(x^2 + y^2\right)$ and $w(x,y) = -\sin(x+y)$ on $(0, 3\pi/4] \times (0, 3\pi/4]$:

$$\int_0^{3\pi/4} \int_0^{3\pi/4} \ln\left(x^2 + y^2\right) \sin(x+y) dx dy \leq \ln \frac{9\pi^2}{8} \int_0^{3\pi/4} \int_0^{3\pi/4} \sin(s+t) ds dt.$$

A similar remark works in the case of Theorem *3.3*.

4. An application to Riemann-Stieltjes integral of several variables

The Riemann-Stieltjes integrals provide a unified approach to the theory of random variables and have proved useful in many fields like stochastic calculus and statistical inference. In analogy with integration over $\mathbb{R}$, the $2d$-monotone functions $f : A \to \mathbb{R}$ give rise to Riemann-Stieltjes integrals. One first defines the integral of a characteristic function of a subinterval $[a,b] \times [c,d] \subset A$ by the formula

$$\iint_{\mathbb{R}^2} \chi_{[a,b]\times[c,d]} df(x,y) = f(a,c) - f(a,d) - f(b,c) + f(b,d),$$

and then extends this formula by linearity and positivity to the linear space $\mathcal{S}t(A)$ of step functions, that is, to the linear combinations of characteristic functions of bounded intervals included in A. Thus

$$\left|\iint_A h(x,y)df(x,y)\right| \leq [f;A]\cdot\|h\|_\infty$$

for every $h \in \mathcal{S}t\,(A)$. Since the elements of $C_c(A)$ (that is, the space of all real-valued continuous functions with compact support included in A) are uniform limits of step functions, one can easily show that $df(x,y)$ is actually a positive Radon measure on A.

Under certain circumstances, the Riemann-Stieltjes integral can be reduced to the Riemann integral. For example, when f is of class C^1 and admits continuous mixed derivatives $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$, then

$$\iint_A h(x,y)df(x,y) = \iint_A h(x,y)\frac{\partial^2 f}{\partial x\partial y}dxdy$$

for all h in $C_c(A)$. The technique of Dirac sequences makes possible to approximate any Stieltjes integral by Riemann integrals of this form. Precisely,

$$\iint_A h(x,y)df(x,y) = \lim_{n\to\infty}\iint_A h(x,y)\frac{\partial^2}{\partial x\partial y}\left(\rho_n * f\right)dxdy,$$

where $\rho \in C_c(\mathbb{R}^2)$ is any nonnegative function such that $\iint_{\mathbb{R}^2}\rho(x,y)dxdy = 1$ and $\rho_n(x,y) = n^2\rho(nx,ny)$ for $n \geq 1$. See Willem [15], Théorème 11.14, p. 57, for details.

Taking into account the above discussion, one can restate Theorem 3.2 as a formula of integration by parts for the Riemann-Stieltjes integral:

Theorem 4.4. *Suppose that f and g are two real-valued functions defined on $[a,b]\times[c,d]$ that fulfill the following conditions: (i) f is continuously differentiable and $2d$-monotone; (ii) g is absolutely continuous in the sense of Carathéodory. Then f is integrable with respect to g and*

$$\int_a^b\int_c^d f(x,y)dg(x,y) = f(b,d)g(b,d)$$
$$-\int_a^b \frac{\partial f(x,d)}{\partial x}g(x,d)dx - \int_c^d\frac{\partial f(b,y)}{\partial y}g(b,y)dy + \int_a^b\int_c^d g(x,y)df(x,y).$$

For $g(x,y) = (x - \lfloor x\rfloor + 1)\,(y - \lfloor y\rfloor + 1)$, Theorem 4.4 yields a formula relating double sums and double integrals

Corollary 4.1. *If $f(x,y)$ is a continuously differentiable function that admits a continuous second order partial derivative $\frac{\partial^2 f}{\partial x\partial y}$ in the rectangle $[a,b]\times[c,d]$, where $a,b,c,d\in\mathbb{Z}$, then*

$$\sum_{a<m\leq b}\sum_{c<n\leq d} f(m,n) = \int_a^b\int_c^d f(x,y)dydx + \int_a^b\int_c^d\frac{\partial f(x,y)}{\partial x}\left(x-\lfloor x\rfloor\right)dydx$$
$$+\int_a^b\int_c^d\frac{\partial f(x,y)}{\partial y}\left(y-\lfloor y\rfloor\right)dydx + \int_a^b\int_c^d\frac{\partial^2 f(x,y)}{\partial x\partial y}\left(x-\lfloor x\rfloor\right)\left(y-\lfloor y\rfloor\right)dydx.$$

Here $\lfloor t \rfloor$ denotes the largest integer not greater than t.

A slightly more restrictive version of Corollary 4.1 was first noticed by V. V. Rane (see [11], Corollary 2), who derived it from the Euler-Maclaurin formula.

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ACADEMY OF ROMANIAN SCIENTISTS
SPLAIUL INDEPENDENŢEI 54
BUCHAREST 050085, ROMANIA
AND UNIVERSITY OF CRAIOVA, CRAIOVA, ROMANIA
E-mail address: cpniculescu@gmail.com